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TECHNICAL REPORT
WVTRR-6007

ELASTIC-PLASTIC ANALYSIS OF A CYLINDRICAL TUBE

BY
R. E. WEIGLE
MARCH - 1960

D. A. PROJECT NO. 5W01-01-033

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LIST OF SYMBOLS

σ_z = Longitudinal Stress

σ_r = Radial Stress

σ_θ = Tangential Stress

σ_r^e = Radial Stress in Elastic Region

σ_r^p = Radial Stress in Plastic Region

σ_0 = Yield Stress in Simple Tension

r, θ, z = Coordinate Axes

u = Radial Displacement

ϵ_r = Radial Strain

ϵ_θ = Tangential Strain

ϵ_z = Longitudinal Strain

a = Internal (bore) Radius

b = External Radius

ρ = Radius to Elastic-Plastic Interface

J_2 = Invariant of the Stress Deviation

k = Yield Stress in Simple Shear

S_{ij} = Stress Deviation

E = Young's Modulus

ν = Poisson Ratio

p = Internal Pressure

p^* = Pressure for Incipient Plastic Flow

p^{**} = Pressure for Complete Plasticity

α = b^2/a^2

β = ρ^2/a^2

γ = $\frac{4k^2}{\sigma_r^2}$

δ = $\frac{4k^2}{p^2}$

ABSTRACT

This report contains an elastic-plastic analysis of a cylindrical tube subjected to an internal pressure P of sufficient magnitude to cause plastic deformation. The Mises yield condition $J_2 = k^2$ is assumed. The tube is considered to be a perfectly plastic material. Results are presented in graphical form for the open-end tube, since the equations expressing the stresses as functions of the yield stress in simple shear (k) and the tube radius cannot be obtained in closed form. Comparison with the plane strain case is made for a wall ratio $\frac{b}{a} = 2$.

DO NOT REMOVE THIS ABSTRACT FROM THE REPORT

CONCLUSIONS

The Mises yield criterion more accurately predicts the plastic behaviour of the pressurized tube than the Tresca yield criterion. In predicting the magnitude and distribution of the residual stresses occurring in the tube, as a result of plastic deformation, the Mises yield criterion should be employed to obtain more realistic values, particularly where the design of tubes having a previous history of plastic deformation is required

From the comparison of the two cases $(\sigma_z = 0)$ and $(\epsilon_z = 0)$ we also conclude that for the tube having longitudinal stresses of a tensile nature, a larger value of internal pressure is required to induce complete plasticity in the tube.

For tubes having small wall ratios of the order of $b/a = 1.2$ and large diameters, the magnitude of the σ_z stress can be quite large. One could then reasonably expect that tubes experiencing large tensile longitudinal stresses will be capable of carrying higher pressures without subsequent plastic deformation.

Further analysis is required in order to verify this supposition and for this purpose an investigation of the closed-end tube is planned.

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INTRODUCTION

A great deal of interest has been shown in the analysis of the elastic-plastic behaviour of thick-walled tubes because of the application of these results to the autofrettage process of cannon.

The theoretical analysis contained in this report was initiated in order to evaluate certain discrepancies reported between experimental and theoretical values for the fully plastic tube as recorded in Ref. 1. The analysis herein indicates conclusively that an erroneous theoretical analysis for the Mises yield criterion was developed. Good agreement was obtained between the theoretically predicted values based upon the present investigation and the experimental results reported in Reference 1.

The determination of the elastic-plastic behaviour of pressurized thick-walled tubes has been investigated by many authors * and many solutions have been obtained. Of course, each solution depends upon the assumptions which must be made regarding the stress-strain relations, yield criterion, end conditions, compressibility, incompressibility, etc.

The following assumptions were made for the problem investigated and reported herein:

1. The tube is subjected to an internal pressure only.

2. The tube material is homogeneous, isotropic, compressible and behaves as a perfectly plastic material, that is, once the material at a point has yielded, deformation will continue with no further increase in load.

3. The stress along the longitudinal axis of the tube vanishes; that is, $\sigma_z = 0$.

* See Reference 1

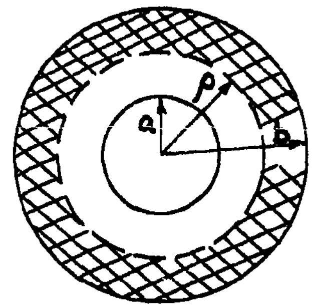
4. The displacement in a radial direction is a function of r alone and does not depend upon z .

5. The condition of constrained plastic deformation exists, that is, unrestricted plastic flow is not permitted.

Through-out the analysis, a right-handed, orthogonal coordinate system (r, θ, z) is employed.

FORMULATION

Consider the accompanying sketch illustrating the tube cross-section, where a, ρ and b denote the internal (bore), elastic-plastic interface and external radii, respectively. Note that the shaded area corresponds to the portion of the tube material which is still elastic. Therefore the elastic region lies in the range $\rho \leq r \leq b$



and the plastic region corresponds to the range $a \leq r \leq \rho$

In the elastic portion of the tube, from the mathematical theory of elasticity, we have the following stress-strain relations:

$$\epsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E}\left[\sigma_r - \nu \sigma_\theta\right] \tag{1}$$

$$\epsilon_\theta = \frac{u}{r} = \frac{1}{E}\left[\sigma_\theta - \nu \sigma_r\right] \tag{2}$$

$$\epsilon_z = \frac{\nu}{E}\left[\sigma_r + \sigma_\theta\right] \tag{3}$$

or solving the above for the stresses we obtain:

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial r} + \nu \frac{u}{r}\right] \tag{4}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu \frac{\partial u}{\partial r}\right] \tag{5}$$

$$\sigma_z = 0 \tag{6}$$

The equilibrium equation is

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{7}$$

Substituting (4), (5) and (6) into (7) we obtain the following ordinary differential equation, since we have already specified that $u = f(r)$ only:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{8}$$

This has a solution of the form

$$u = Ar + \frac{B}{r} \tag{9}$$

where the constants A and B must be determined from the boundary conditions. For the elastic region $(\rho \leq r \leq b)$ these boundary conditions are, using the Mises criterion of yielding,

$$r = \rho \rightarrow \sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2 = 3k^2 \tag{10}$$

$$r = b \rightarrow \sigma_r = 0 \tag{11}$$

Note that the Mises yield criterion

$$J_2 = k^2 = \frac{1}{2} S_{ij} S_{ij} \tag{12}$$

when written in terms of the principal stresses reduces to (10) where k is the yield stress in simple shear and S_{ij} are the stress deviators.

Solving for A and B, we write σ_r and σ_θ in terms of A and B, by use of (4), (5) and (9), and have

$$\sigma_r = \left[\frac{E}{1-\nu^2}\right]\left[(1+\nu)A - (1-\nu)\frac{B}{r^2}\right] \tag{13}$$

$$\sigma_\theta = \left[\frac{E}{1-\nu^2}\right]\left[(1+\nu)A + (1-\nu)\frac{B}{r^2}\right] \tag{14}$$

Using the boundary condition (10), (13) and (14) we obtain

$$\left[\frac{E}{1-\nu^2}\right]^2\left[(1+\nu)^2A^2 - 2(1+\nu)(1-\nu)\frac{AB}{\rho^2} + (1-\nu)^2\frac{B^2}{\rho^4}\right] - \left[\frac{E}{1-\nu^2}\right]^2\left[(1+\nu)^2A^2 - (1-\nu)^2\frac{B^2}{\rho^4}\right]$$

$$+\left[\frac{E}{1+\nu^2}\right]^2\left[(1+\nu)^2A^2 + 2(1+\nu)(1-\nu)\frac{AB}{\rho^2} + (1-\nu)^2\frac{B^2}{\rho^4}\right] = 3k^2 \tag{15}$$

which reduces to

$$(1+\nu)^2A^2 + 3(1-\nu)^2\frac{B^2}{\rho^4} = \frac{3(1-\nu^2)^2}{E^2}k^2 \tag{16}$$

Applying (11) and (13) we have

$$\left[\frac{E}{1-\nu^2}\right]\left[(1+\nu)A - (1-\nu)\frac{B}{b^2}\right] = 0 \tag{17}$$

whence

$$A = \frac{1-\nu}{1+\nu}\,\frac{B}{b^2} \tag{18}$$

Using (16) and (18) we find

$$A = k\left[\frac{1-\nu}{Eb^2}\right]\left[\frac{1}{3b^4} + \frac{1}{\rho^4}\right]^{-\frac{1}{2}} \tag{19}$$

$$B = k\left[\frac{1+\nu}{Eb^2}\right]\left[\frac{1}{3b^4} + \frac{1}{\rho^4}\right]^{-\frac{1}{2}} \tag{20}$$

Therefore, from (13), (14), (19) and (20) we have for the elastic region $(\rho \leq r \leq b)$ the following expressions for the stresses:

$$\sigma_r = k\left[\frac{1}{3b^4} + \frac{1}{\rho^4}\right]^{-\frac{1}{2}}\left[\frac{1}{b^2} - \frac{1}{r^2}\right] \tag{21}$$

$$\sigma_\theta = k\left[\frac{1}{3b^4} + \frac{1}{\rho^4}\right]^{-\frac{1}{2}}\left[\frac{1}{b^2} + \frac{1}{r^2}\right] \tag{22}$$

To obtain the pressure, P^*, for incipient plastic flow we know that for

$$r = a \longrightarrow \sigma_r = -P \tag{23}$$

and (21) will be a maximum for $r = \rho = a$. Therefore

$$P^* = -k\left[\frac{1}{3b^4} + \frac{1}{a^4}\right]^{-\frac{1}{2}}\left[\frac{1}{b^2} - \frac{1}{a^2}\right] \tag{24}$$

To obtain the stresses in the plastic region $(a \leq r \leq \rho)$, we note that the equilibrium equation (7) must be satisfied. Writing (12) in terms of the stresses we have

$$\sigma_\theta^2 - \sigma_r\sigma_\theta + \sigma_r^2 - 3k^2 = 0 \tag{25}$$

which, when solved for σ_θ yields

$$\sigma_\theta = \frac{1}{2}\left[\sigma_r \pm (12k^2 - 3\sigma_r^2)^{\frac{1}{2}}\right] \tag{26}$$

To determine the proper sign of the radical, we note that in the elastic region, from (21) and (22)

$$2\sigma_\theta - \sigma_r = k\left[\frac{1}{3b^4} + \frac{1}{\rho^4}\right]^{-\frac{1}{2}}\left[\frac{1}{b^2} + \frac{3}{r^2}\right] > 0 \tag{27}$$

which must be valid at the elastic-plastic interface. Thus, from (26), we write

$$2\sigma_\theta - \sigma_r = \pm\left[12k^2 - 3\sigma_r^2\right] > 0 \tag{28}$$

and hence, (28) can only be satisfied if the positive value of the radical is specified. Therefore

$$\sigma_\theta = \frac{1}{2}\left[\sigma_r + \left(12k^2 - 3\sigma_r^2\right)^{\frac{1}{2}}\right] \tag{29}$$

and where the radical appears we will assign a positive value to it.

Substituting (29) into (7) we obtain

$$\frac{d\sigma_r}{dr} = \frac{1}{2r}\left[-\sigma_r + \left(12k^2 - 3\sigma_r^2\right)^{\frac{1}{2}}\right] \tag{30}$$

or

$$\int \frac{d\sigma_r}{-\sigma_r + \left(12k^2 - 3\sigma_r^2\right)^{\frac{1}{2}}} = \frac{1}{2}\int \frac{dr}{r} + C_0 \tag{31}$$

To integrate the left side of (31) we employ a change of variables as follows: Let

$$\sigma_r = -\frac{2k}{\left(\xi^2 + 1\right)^{\frac{1}{2}}} \tag{32}$$

and the integral becomes

$$\int \frac{\xi \, d\xi}{(\xi^2+1)(1+\sqrt{3}\,\xi)} \tag{33}$$

which can be integrated directly. Making use of the results obtained from (33) and writing (31) in terms of σ_r, k and r we get

$$\ln \gamma - \ln\left[1+\left|(3\gamma-3)^{\frac{1}{2}}\right|\right]^2 + 2\sqrt{3}\tan^{-1}\left|(\gamma-1)^{\frac{1}{2}}\right| - 4\ln r = C_0 \tag{34}$$

where

$$\gamma = \frac{4k^2}{\sigma_r^2} \tag{35}$$

We also define the following:

$$\alpha = \frac{b^2}{a^2}\ ,\ \beta = \frac{\rho^2}{a^2}\ ,\ \delta = \frac{4k^2}{p^2} \tag{36}$$

The integration constant C_0, can now be evaluated from the continuity condition at the elastic-plastic interface, that is, for

$$r = \rho \longrightarrow \sigma_r^e = \sigma_r^p \tag{37}$$

From (21) and (36) we find

$$\sigma_r^e\Big|_{r=\rho} = \frac{k\sqrt{3}\,(\beta-\alpha)}{\left[\beta^2+3\alpha^2\right]^{\frac{1}{2}}} = \sigma_r^p\Big|_{r=\rho} \tag{38}$$

Therefore, from (35) and (38)

$$\gamma\Big|_{r=\rho} = \frac{4(\beta^2+3\alpha^2)}{3(\beta-\alpha)^2} \tag{39}$$

In view of (37) and using (39) in (34) we find

$$C_0 = \ln\frac{4(\beta^2+3\alpha^2)}{3(\beta-\alpha)^2} - \ln\left[1+\left|\left\{\frac{4(\beta^2+3\alpha^2)}{(\beta-\alpha)^2}-3\right\}^{\frac{1}{2}}\right|\right]^2$$
$$+ 2\sqrt{3}\tan^{-1}\left|\left\{\frac{4(\beta^2+3\alpha^2)}{3(\beta-\alpha)^2}-1\right\}^{\frac{1}{2}}\right| - 4\ln\rho \tag{40}$$

Since $\alpha \geq \beta$ we may re-write (40) and obtain

$$C_0 = \ln\frac{\beta^2+3\alpha^2}{12\alpha^2} + 2\sqrt{3}\tan^{-1}\left[\frac{\beta+3\alpha}{\sqrt{3}(\alpha-\beta)}\right] - 4\ln\rho \tag{41}$$

Therefore, (34) together with (41) yields the following transcendental equation, which must be solved by numerical methods to obtain the value of σ_r in the plastic region of the tube. This equation is valid throughout the tube.

$$\ln\gamma - \ln\left[1+\left|(3\gamma-3)^{\frac{1}{2}}\right|\right]^2 + 2\sqrt{3}\tan^{-1}\left|(\gamma-1)^{\frac{1}{2}}\right|$$
$$+ \ln\frac{12\alpha^2\beta^2}{\beta^2+3\alpha^2} - 2\sqrt{3}\tan^{-1}\left[\frac{\beta+3\alpha}{\sqrt{3}(\alpha-\beta)}\right] - 4\ln\frac{r}{a} = 0 \tag{42}$$

We must also determine the value of the internal pressure which corresponds to a given value of ρ. This relationship may be determined from the boundary condition for

$$r=a \longrightarrow \sigma_r = -p \tag{43}$$

From (35) and (36), together with the boundary condition (43), we have

$$\gamma \big|_{r=a} = \delta \tag{44}$$

Evaluating (42) at the boundary $r=a$ and using the relationship (44), we obtain the equation relating p to ρ, which follows

$$\ln \delta - \ln\left[1+\left|(3\delta-3)^{\frac{1}{2}}\right|\right]^2 + 2\sqrt{3}\tan^{-1}\left|(\delta-1)^{\frac{1}{2}}\right|$$

$$+\ln\frac{12\alpha^2\beta^2}{\beta^2+3\alpha^2} - 2\sqrt{3}\tan^{-1}\left[\frac{\beta+3\alpha}{\sqrt{3}(\alpha-\beta)}\right] = 0 \tag{45}$$

For contained plastic deformation we cannot permit the value of the internal pressure to exceed p^{**}. This is the maximum pressure which can be applied and corresponds to the condition for complete plasticity in the tube; that is, when $\rho = b$. Therefore, to determine p^{**} we let $\alpha = \beta$ and $\delta = \delta^{**}$, where

$$\delta^{**} = \frac{4k^2}{(p^{**})^2} \tag{46}$$

In view of the above, (45) becomes

$$\ln \delta^{**} - \ln\left[1+\left|(3\delta^{**}-3)^{\frac{1}{2}}\right|\right]^2 + 2\sqrt{3}\tan^{-1}\left|(\delta^{**}-1)^{\frac{1}{2}}\right|$$

$$+\ln 3\alpha^2 - \sqrt{3}\,\pi = 0 \tag{47}$$

which can be solved by numerical methods for δ^{**} and hence, P^{**}.

In order to have plastic deformation occur, the internal pressure must lie within the following range

$$P^{*} \leq P \leq P^{**} \tag{48}$$

For the above range of P, equation (45) is valid in the plastic region $(a \leq r \leq \rho)$ and equations (21) and (22) are valid in the elastic region $(\rho \leq r \leq b)$.

To determine σ_θ in the plastic region, one must use (29) together with the values of σ_r as determined from (45).

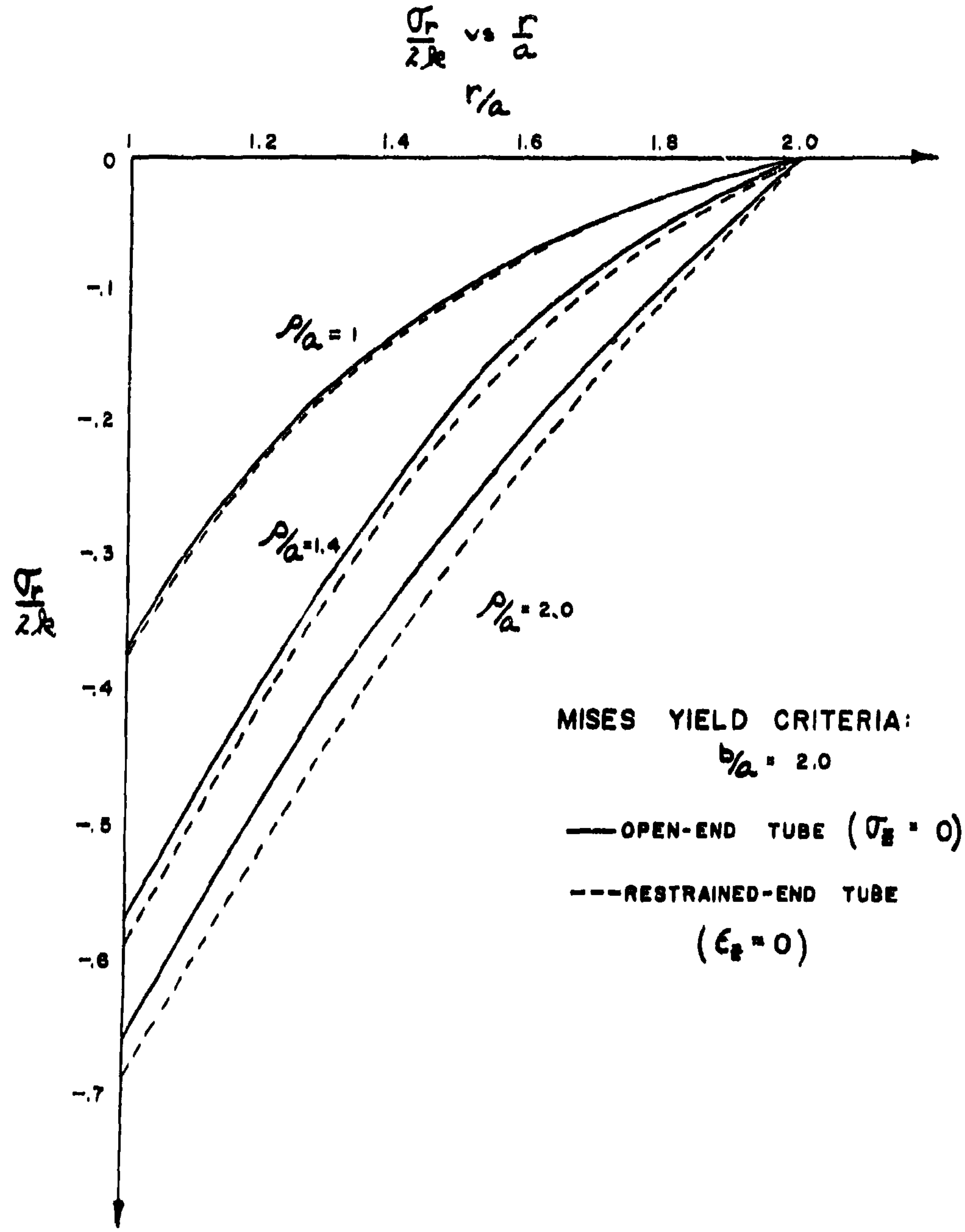
If $P < P^{*}$ then the tube behavior is completely elastic and the well known Lamé equations apply. These are given below

$$\sigma_r = \left[\frac{P}{\alpha - 1}\right]\left[1 - \frac{b^2}{r^2}\right] \tag{49}$$

$$\sigma_\theta = \left[\frac{P}{\alpha - 1}\right]\left[1 + \frac{b^2}{r^2}\right] \tag{50}$$

RESULTS

Graphical results are given in dimensionless form for $b/a = 2.0$ and several values of ρ/a for the case $(\sigma_z = 0)$. For this particular case a direct comparison of the values of $\sigma_r/2k$ vs. r/a is made with similar results reported by Prager and Hodge for the case $(\epsilon_z = 0)$ where $b/a = 2.0$. As can be seen from Figure I, for any given internal pressure the amount of plastic deformation will be larger for the case $(\sigma_z = 0)$. Figure II shows the agreement obtained between theoretically predicted and experimentally determined values of the dimensionless pressure, p/σ_0 and various wall ratios for both the Mises and Tresca yield conditions.



σr/2k vs r/a
r/a
1
1.2
1.4
1.6
1.8
2.0
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
σr/2k
ρ/a = 1
ρ/a = 1.4
ρ/a = 2.0
MISES YIELD CRITERIA:
b/a = 2.0
——OPEN-END TUBE (σz = 0)
---RESTRAINED-END TUBE
(εz = 0)

FIG. 1

$\frac{P}{\sigma_0}$ vs $\frac{b}{a}$

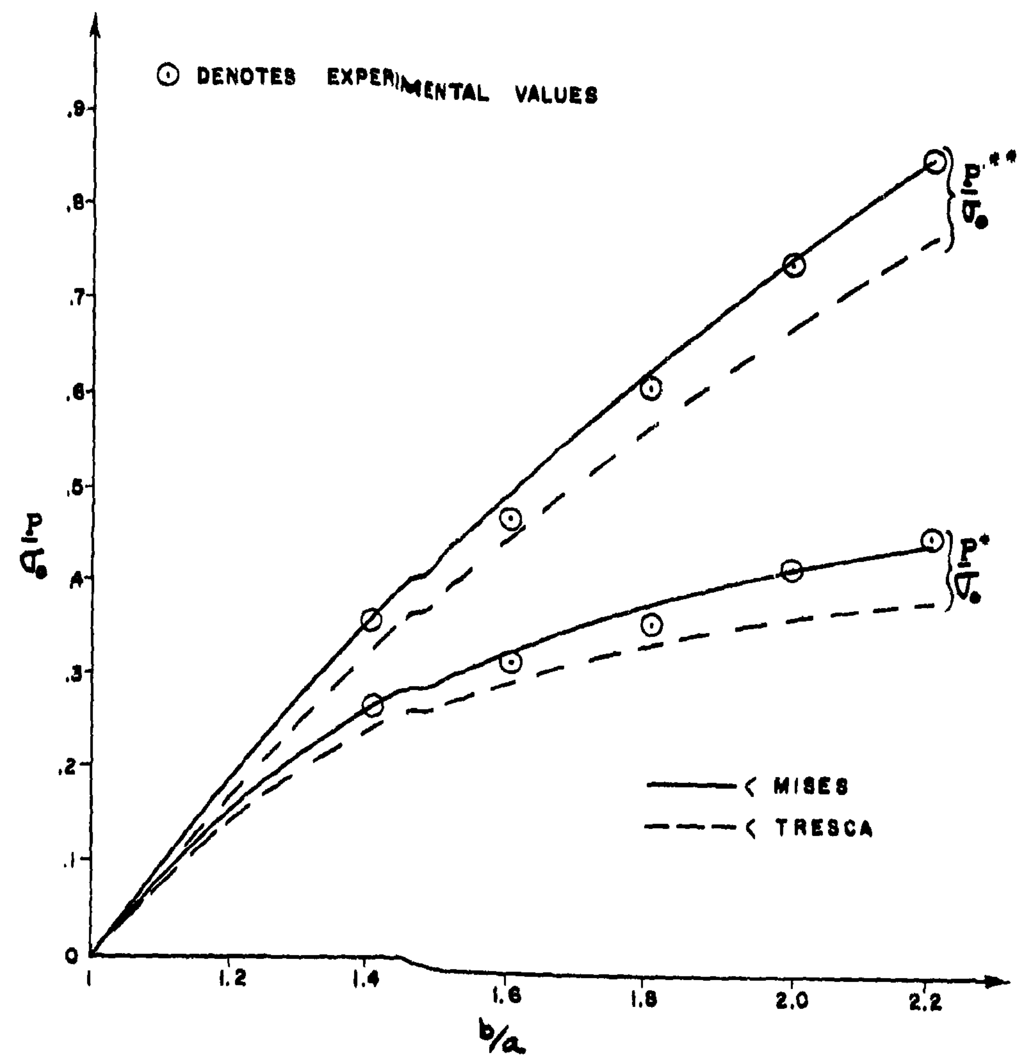


FIG II

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